(a) $e^x \geq x+1, \forall x \in \mathbb{R}$

The equality holds if and only if $x=0$

Pf: · When $x=0$, L.H.S = R.H.S = 1.

· When $x>0$, since $(e^x)' = e^x$,

by MVT, $\exists\ c \in (0,x)$ s.t.

$$e^x - e^0 = e^c(x-0) > x, \text{ i.e.}$$

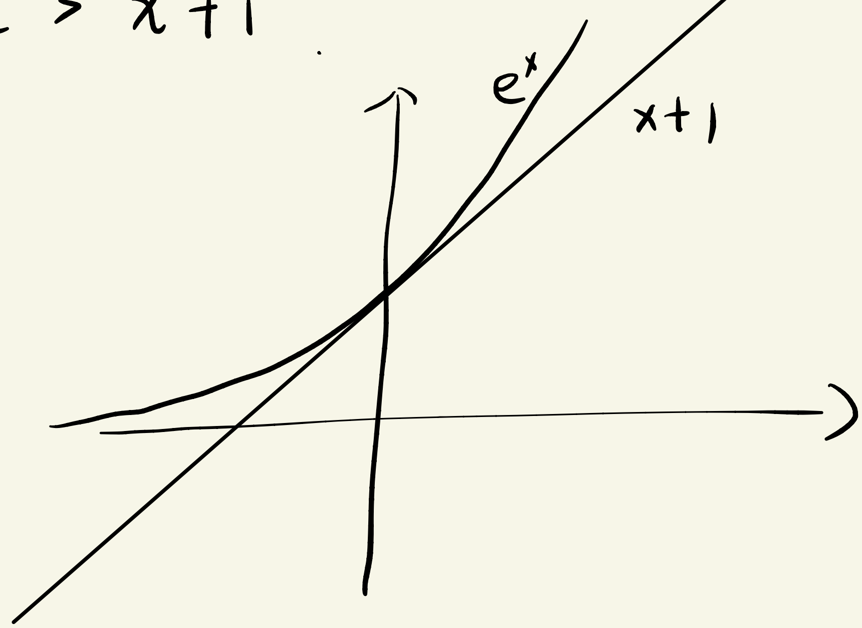
$$e^x > x+1.$$

· When $x<0$, by MVT,

$\exists\ c \in (x,0)$ s.t.

$$e^0 - e^x = e^c(0-x) < -x, \text{ i.e.}$$

$$e^x > x+1.$$



$\square$

(b) $-x \le \sin x \le x$, $\forall x \in \mathbb{R}$

- When $x = 0$, $\sin x = 0$
- When $x > 0$, since $(\sin x)' = \cos x$,

  by MVT, $\exists\, c \in (0, x)$ s.t.

  $$\sin x - \sin 0 = \cos c\,(x - 0)$$

  Since $-1 \le \cos c \le 1$,

  $$-x \le \sin x \le x.$$

- When $x < 0$, by MVT,

  $\exists\, c \in (x, 0)$ s.t.

  $$\sin 0 - \sin x = \cos c\,(0 - x)$$

  Since $-1 \le \cos c \le 1$,

  $$x \le -\sin x \le -x, \text{ i.e.,}$$

  $$x \le \sin x \le -x$$

$\square$

(c) If $\alpha > 1$, $(x+1)^\alpha \geq \alpha x + 1$ for all $x \geq -1$ with equality if and only if $x = 0$.

Pf: · When $x = 0$, L.H.S. = R.H.S. = 1.

· When $x > 0$, since $\left((x+1)^\alpha\right)' = \alpha(x+1)^{\alpha-1}$

by MVT, $\exists$ $c \in (0, x)$ s.t.

$$(x+1)^\alpha - 1 = \alpha(c+1)^{\alpha-1}(x-0) > \alpha x, \text{ i.e.,}$$

$$(x+1)^\alpha > \alpha x + 1.$$

· When $-1 \leq x < 0$, by MVT,

$\exists$ $c \in (x, 0)$ s.t.

$$1 - (x+1)^\alpha = \alpha(c+1)^{\alpha-1}(-x) < -\alpha x, \text{ i.e.,}$$

$$(x+1)^\alpha > \alpha x + 1.$$

$\square$

(d) If $a, b > 0$ and $0 < \alpha < 1$, then

$$a^{\alpha} b^{1-\alpha} \leq \alpha a + (1-\alpha) b$$ with

equality if and only if.

Pf: This inequality is equivalent to

$$\left(\frac{a}{b}\right)^{\alpha} \leq \alpha\left(\frac{a}{b}\right) + (1-\alpha).$$

It suffices to show

if $x > 0$ and $0 < \alpha < 1$, then

$x^{\alpha} \leq \alpha x + (1-\alpha)$ with equality

if and only if $x = 1$.

- When $x = 1$, L.H.S. = R.H.S. = 1.
- When $x > 1$, since $(x^{\alpha})' = \alpha x^{\alpha-1}$,

by MVT, $\exists\ c \in (1, x)$ s.t.

$$x^{\alpha} - 1 = \alpha c^{\alpha-1}(x-1)$$

$$< \alpha(x-1)$$, i.e.

$$x^{\alpha} < \alpha x + (1-\alpha).$$

- When $0 < x < 1$, by MVT,

$\exists$ $c \in (x, 1)$ s.t.

$$1 - x^{\alpha} = \alpha c^{\alpha-1}(1-x)$$

$$> \alpha(1-x) \quad , \quad \text{i.e.}$$

$$x^{\alpha} < \alpha x + (1-\alpha) .$$

$\square$